# Chapter 11
# Car-Following Models based on Driving Strategies

*Ideas are like children: you always love your own the most.*
Lothar Schmidt

**Abstract** The models introduced in this chapter are derived from assumptions about real driving behavior such as keeping a "safe distance" from the leading vehicle, driving at a desired speed, or preferring accelerations to be within a comfortable range. Additionally, kinematical aspects are taken into account, such as the quadratic relation between braking distance and speed. We introduce two examples: The simplified Gipps model, and the Intelligent Driver Model. Both models use the same input variables as the sensors of *adaptive cruise control* (ACC) systems, and produce a similar driving behavior. Characteristics that are specific to the human nature, like erroneous judgement, reaction time, and multi-anticipation, are discussed in the next chapter.

## 11.1 Model Criteria

The models introduced in this chapter are formally identical to the minimal models presented in the previous chapter. They are defined by an acceleration function $a_{\text{mic}}$ (see Eq. (10.3)) or a speed function $v_{\text{mic}}$ (see Eq. (10.7)). In contrast to the minimal models, the acceleration or speed functions encoding the driving behavior should at least model the following aspects:

1. The acceleration is a strictly decreasing function of the speed. Moreover, the vehicle accelerates towards a *desired speed* $v_0$ if not constrained by other vehicles or obstacles:

$$\frac{\partial a_{\text{mic}}(s,v,v_l)}{\partial v} < 0, \quad \lim_{s\to\infty} a_{\text{mic}}(s,v_0,v_l) = 0 \quad \text{for all } v_l. \tag{11.1}$$

2. The acceleration is an increasing function of the distance $s$ to the leading vehicle:

$$\frac{\partial a_{\text{mic}}(s,v,v_l)}{\partial s} \geq 0, \quad \lim_{s\to\infty} \frac{\partial a_{\text{mic}}(s,v,v_l)}{\partial s} = 0 \quad \text{for all } v_l. \tag{11.2}$$

The inequality becomes an equality if other vehicles or obstacles (including "virtual" obstacles such as the stopping line at a red traffic light) are outside the interaction range and therefore do not influence the driving behavior. This defines the *free-flow acceleration*

$$a_{\text{free}}(v) = \lim_{s\to\infty} a_{\text{mic}}(s,v,v_l) = \geq a_{\text{mic}}(s,v,v_l). \tag{11.3}$$

3. The acceleration is an increasing function of the speed of the leading vehicle. Together with requirement (1), this also means that the acceleration decreases (the deceleration increases) with the speed of approach to the lead vehicle (or obstacle):

$$\frac{\partial \tilde{a}_{\text{mic}}(s,v,\Delta v)}{\partial \Delta v} \leq 0 \ \text{ or } \ \frac{\partial a_{\text{mic}}(s,v,v_l)}{\partial v_l} \geq 0, \ \lim_{s\to\infty} \frac{\partial a_{\text{mic}}(s,v,v_l)}{\partial v_l} = 0. \tag{11.4}$$

   Again, the equality holds if other vehicles (or obstacles) are outside the interaction range.
4. A minimum gap (bumper-to-bumper distance) $s_0$ to the leading vehicle is maintained (also during a standstill). However, there is no backwards movement if the gap has become smaller than $s_0$ by past events:

$$a_{\text{mic}}(s,0,v_l) = 0 \quad \text{for all } v_l \geq 0, \ s \leq s_0. \tag{11.5}$$

By virtue of relation (10.11), these requirements (or *plausibility conditions*) for the acceleration function naturally imply conditions for the speed function $v_{\text{mic}}$ of models formulated in terms of coupled maps.

A car-following model meeting these requirements is *complete* in the sense that it can consistently describe all situations that may arise in single-lane traffic. Particularly, it follows that (i) all vehicle interactions are of finite reach, (ii) following vehicles are not "dragged along",

$$a_{\text{mic}}(s,v,v_l) \leq a_{\text{mic}}(\infty,v,v_l') = a_{\text{free}}(v) \quad \text{for all } s,v,v_l, \text{ and } v_l', \tag{11.6}$$

and (iii) an equilibrium speed $v_e(s)$ exists, which has the properties already postulated for the optimal-speed function (10.20):

$$v_e'(s) \geq 0, \quad v_e(0) = 0, \quad \lim_{s\to\infty} v_e(s) = v_0. \tag{11.7}$$

This means that the model possesses a unique steady-state flow-density relation, i.e., a fundamental diagram.[1]

These conditions are necessary but not sufficient. For example, when in the car-following regime (steady-state congested traffic), the time gap to the leader has to remain within reasonable bounds (say, between 0.5 s and 3 s). Furthermore, the ac-

[1] If one were to weaken condition (11.1) to $\partial a_{\text{mic}}/\partial v \leq 0$, it is possible to formulate models that do *not* have a fundamental diagram. Such models are proposed in the context of B. Kerner's *three-phase theory*.

celeration has to be constrained to a "comfortable" range (e.g., $\pm 2\,\mathrm{m/s^2}$), or at least, to physically possible values. Particularly, when approaching the leading vehicle, the quadratic relation between braking distance and speed has to be taken into account. Finally, any car-following model should allow instabilities and thus the emergence of "stop-and-go" traffic waves, but should not produce accidents, i.e., negative bumper-to-bumper gaps $s < 0$.[2]

Which of the car-following models introduced in Chapter 10 satisfy the conditions (11.1) – (11.5)?

## 11.2 Gipps' Model

Gipps' model presented here is a modified version of the one described in his original publication. It is simplified, but conceptually unchanged. Although it produces an unrealistic acceleration profile, this model is probably the simplest complete and accident-free model that leads to accelerations within a realistic range.

### *11.2.1 Safe Speed*

Accidents are prevented in the model by introducing a "safe speed" $v_{\text{safe}}(s, v_l)$, which depends on the distance to and speed of the leading vehicle. It is based on the following assumptions:

1. Braking maneuvers are always executed with constant deceleration $b$. There is no distinction between comfortable and (physically possible) maximum deceleration.
2. There is a constant "reaction time" $\Delta t$.
3. Even if the leading vehicle suddenly decelerates to a complete stop (worst case scenario), the distance gap to the leading vehicle should not become smaller than a minimum gap $s_0$.[3]

Condition 1 implies that the *braking distance* that the leading vehicle needs to come to a complete stop is given by

$$\Delta x_l = \frac{v_l^2}{2b}.$$

[2] Traffic-flow models are meant to describe *normal* conditions, while accidents are almost always caused by *exceptional* driving mistakes that are not part of normal driving behavior and thus not part of the intended scope of the model.

[3] This condition is not present in the original paper, but is necessary to ensure an accident-free model in the presence of numerical errors arising from discretization.

From condition 2 it follows that, in order to come to a complete stop, the driver of the considered vehicle needs not only his or her braking distance $v^2/(2b)$, but also an additional *reaction distance* $v\Delta t$ travelled during the reaction time.[4] Consequently, the *stopping distance* is given by

$$\Delta x = v\Delta t + \frac{v^2}{2b}. \tag{11.8}$$

Finally, condition 3 is satisfied if the gap $s$ exceeds the required minimum final value $s_0$ by the difference $\Delta x - \Delta x_l$ between the stopping distance of the considered vehicle and the breaking distance of the leader:

$$s \geq s_0 + v\Delta t + \frac{v^2}{2b} - \frac{v_l^2}{2b}. \tag{11.9}$$

The speed $v$ for which the equal sign holds (the highest possible speed) defines the "safe speed"

$$v_{\text{safe}}(s, v_l) = -b\Delta t + \sqrt{b^2\Delta t^2 + v_l^2 + 2b(s - s_0)}. \tag{11.10}$$

### 11.2.2 Model Equation

The simplified Gipps' model is defined as an iterated map with the "safe speed" (11.10) as its main component:

$$\boxed{v(t+\Delta t) = \min\left[v + a\Delta t, v_0, v_{\text{safe}}(s, v_l)\right] \quad \text{Gipps' model.}} \tag{11.11}$$

This model equation reflects the following properties:

- The simulation update time step is equal to the reaction time $\Delta t$.
- If the current speed is greater than $v_{\text{safe}} - a\,\Delta t$ or $v_0 - a\,\Delta t$, the vehicle will reach the minimum of $v_0$ and $v_{\text{safe}}$ during the next time step.[5]
- Otherwise the vehicle accelerates with constant acceleration $a$ until either the safe speed or the desired speed is reached.

### 11.2.3 Steady-State Equilibrium

The homogeneous steady state implies $v(t+\Delta t) = v_l = v$, thus

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[4] In contrast to the original publication, we assume the speed to be constant within the reaction time.

[5] Strictly speaking, this means that deceleration $(v - v_{\text{safe}})/\Delta t$ is not restricted to $b$. In multi-lane simulations, it can be greater if another vehicle "cuts in" in front of the considered vehicle.

$$v = \min(v_0, v_{\text{safe}}) = \min\left(v_0, -b\Delta t + \sqrt{b^2\Delta t^2 + v^2 + 2b(s - s_0)}\right),$$

which yields the steady-state speed-gap relation

$$v_e(s) = \max\left[0, \min\left(v_0, \frac{s - s_0}{\Delta t}\right)\right] \tag{11.12}$$

and, assuming constant vehicle lengths $l$, the familiar "triangular" fundamental diagram

$$Q_e(\rho) = \min\left(v_0\rho, \frac{1 - \rho l_{\text{eff}}}{\Delta t}\right), \tag{11.13}$$

where $l_{\text{eff}} = (l + s_0)$. As in the Newell model, the parameter $\Delta t$ can be interpreted in four different ways: (i) As the reaction time introduced in the derivation of $v_{\text{safe}}$, (ii) as the numerical update time step of the actual model equation (11.11), (iii) as a speed adaption time in Eq. (11.11) (at least, if $v(t + \Delta t)$ is restricted by $v_{\text{safe}}$ or $v_0$), or (iv) as the "safety time gap" $(s - s_0)/v_e$ in congested traffic as deduced from the fundamental diagram (11.12).

**Table 11.1** Parameters of the simplified Gipps' model and typical values in different scenarios.

| Parameter | Typical Value Highway | Typical Value City Traffic |
|---|---|---|
| Desired speed $v_0$ | 120 km/h | 54 km/h |
| Adaption/reaction time $\Delta t$ | 1.1 s | 1.1 s |
| Acceleration $a$ | $1.5\,\text{m/s}^2$ | $1.5\,\text{m/s}^2$ |
| Deceleration $b$ | $1.0\,\text{m/s}^2$ | $1.0\,\text{m/s}^2$ |
| Minimum distance $s_0$ | 3 m | 2 m |

### *11.2.4 Model Characteristics*

Unlike the minimal models described in the previous chapter, the Gipps' model is transparently derived from a few basic assumptions and uses parameters that are easy to interpret and assign realistic values (Table 11.1). Furthermore, Gipps' model is – again, in contrast to the minimal models – *robust* in the sense that meaningful results can be produced from a comparatively wide range of parameter values.

**Highway traffic.** The simulation of the highway scenario (Fig. 11.1, left) produces more realistic results than the OVM or the Newell model: The speed field in panel (a) exhibits small perturbations which are caused by vehicles merging from the on-ramp and grow into stop-and-go waves while propagating upstream. The propagation velocity $c_{\text{cong}} = -l_{\text{eff}}/\Delta t$ is constant and of the order of the empirical value

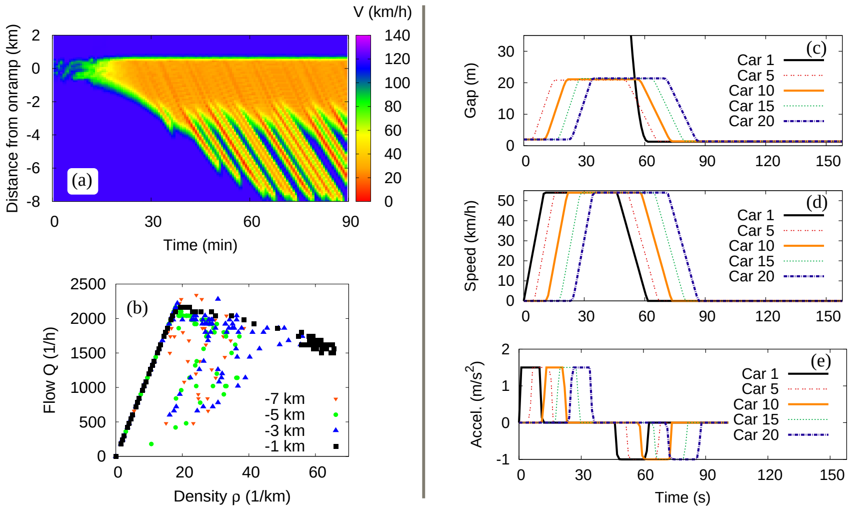


**Fig. 11.1** Fact sheet of Gipps' model (11.11),(11.10). Simulation of the two standard scenarios "highway" (left) and "city traffic" (right) with the parameter values listed in Table 11.1. See Chapter 10.5 for a detailed description of the scenarios.

($\approx -15\,\mathrm{km/h}$). Furthermore, the wave length (of the order of $1-1.5\,\mathrm{km}$) is not too far away from the empirical values ($1.5-3\,\mathrm{km}$).

The flow-density diagram in Fig. 11.1(b), obtained from virtual detectors, shows a strongly scattered cloud of data points in the region of congested traffic, i.e., everywhere to the right of the straight line indicating free traffic. Such a wide scattering is in agreement with empirical data (cf. Figs. 4.11 and 4.12). By looking at scatter plots of individual detectors, one observes that detectors that are closer to the bottleneck produce data points that are shifted towards greater densities and closer to the fundamental diagram of steady-state traffic. Moreover, the data points of virtual detectors positioned inside the region of stationary traffic immediately upstream of the bottleneck (solid black squares) lie on the fundamental diagram itself. This apparent density increase near the outflow region of a congestion, also known as *pinch effect*, can be observed empirically. However, the systematic density underestimation, which conspicuously increases with the degree of the scattering of the data points, suggests that the *real* density increase is smaller, or even nonexistent. This means that the pinch effect is essentially a result of data misinterpretation, or, more specifically, by estimating the density with the time mean speed instead of the space mean speed (cf. Section 3.3.1). This interpretation is confirmed by simulation as will be shown in Fig. 11.5(b) below. We draw an important conclusion that is not restricted to Gipps' model (and not even to traffic flow models):

When using empirical data to assert the accuracy and predictive power of models, one has to simulate both the actual traffic dynamics *and* the process of data capture and analysis.

**City traffic.** Compared to the simple models of the previous chapter, the city-traffic scenario (Fig. 11.1, right column) is closer to reality as well. However, the acceleration time-series is unrealistic. By definition, there are only three values for the acceleration: Zero, $a$, and $-b$ (cf. Panel (e)). The resulting driving behavior is excessively "robotic" and the abrupt transitions are unrealistic.

Moreover, Gipps' model does not differentiate between comfortable and maximum deceleration: Assuming that $b$ in Eq. (11.10) denotes the maximum deceleration, the model is accident-free but every braking maneuver is performed *very* uncomfortably with full brakes. On the other hand, when interpreting $b$ as the comfortable deceleration and allowing for heterogeneous and/or multi-lane traffic the model possibly produces accidents if leading vehicles (which might be simulated using different parameters or even different models) brake harder than $b$.

In summary, Gipps' model produces good results in view of its simplicity. Modified versions of this model are used in several commercial traffic simulators. One example of such a modification is *Krauss' model* which essentially is a stochastic version of the Gipps model.

## 11.3 Intelligent Driver Model

The time-continuous *Intelligent Driver Model* (IDM) is probably the simplest complete and accident-free model producing realistic acceleration profiles and a plausible behavior in essentially all single-lane traffic situations.

### *11.3.1 Required Model Properties*

As Gipps' model, the IDM is derived from a list of basic assumptions (*first-principles model*). It is characterized by the following requirements:

1. The acceleration fulfills the general conditions (11.1) – (11.5) for a complete model.
2. The equilibrium bumper-to-bumper distance to the leading vehicle is not less than a "safe distance" $s_0 + vT$ where $s_0$ is a minimum (bumper-to-bumper) gap, and $T$ the (bumper-to-bumper) time gap to the leading vehicle.
3. An *braking strategy!intelligent* controls how slower vehicles (or obstacles or red traffic lights) are approached:

- Under normal conditions, the braking maneuver is "soft", i.e., the deceleration increases gradually to a comfortable value $b$, and decreases smoothly to zero just before arriving at a steady-state car-following situation or coming to a complete stop.
- In a critical situation, the deceleration exceeds the comfortable value until the danger is averted. The remaining braking maneuver (if applicable) will be continued with the regular comfortable deceleration $b$.

4. Transitions between different driving modes (e.g., from the acceleration to the car-following mode) are smooth. In other words, the time derivative of the acceleration function, i.e., the *jerk* $J$, is finite at all times.[6] This is equivalent to postulating that the acceleration function $a_{\text{mic}}(s, v, v_l)$ (or $\tilde{a}_{\text{mic}}(s, v, \Delta v)$) is continuously differentiable in all three variables. Notice that this postulate is in contrast to the action-point models such as the *Wiedemann Model* where acceleration changes are modeled as a series of discrete jumps.
5. The model should be as parsimonious as possible. Each model parameter should describe only one aspect of the driving behavior (which is favorable for model calibration). Furthermore, the parameters should correspond to an intuitive interpretation and assume plausible values.

### 11.3.2 Mathematical Description

The required properties are realized by the following acceleration equation:

$$\boxed{\dot{v} = a\left[1 - \left(\frac{v}{v_0}\right)^{\delta} - \left(\frac{s^*(v, \Delta v)}{s}\right)^2\right] \quad \text{IDM.}} \tag{11.14}$$

The acceleration of the Intelligent Driver Model is given in the form $\tilde{a}_{\text{mic}}(s, v, \Delta v)$ and consists of two parts, one comparing the current speed $v$ to the desired speed $v_0$, and one comparing the current distance $s$ to the desired distance $s^*$. The desired distance

$$s^*(v, \Delta v) = s_0 + \max\left(0, vT + \frac{v\Delta v}{2\sqrt{ab}}\right) \tag{11.15}$$

has an equilibrium term $s_0 + vT$ and a *dynamical term* $v\Delta v/(2\sqrt{ab})$ that implements the "intelligent" braking strategy (see Section 11.3.4).[7]

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[6] Typical values of a "comfortable" jerk are $|J| \leq 1.5\,\text{m/s}^3$.

[7] The maximum condition in Eq. (11.15) ensures that the conditions (11.1) – (11.5) for model completeness hold for all situations. Strictly speaking, this condition violates the postulate of a smooth acceleration function. However, it comes into effect only in two situations: (i) For finite speeds if the leading car is much faster, (ii) for stopped queued vehicles when the queue starts to move. The first situation may arise after a cut-in maneuver of a faster vehicle. Since $s \gg s_0$ for this case, the resulting discontinuity is small. In the second case, the maximum condition prevents an overly sluggish start and the associated discontinuous acceleration profile may even be realistic.

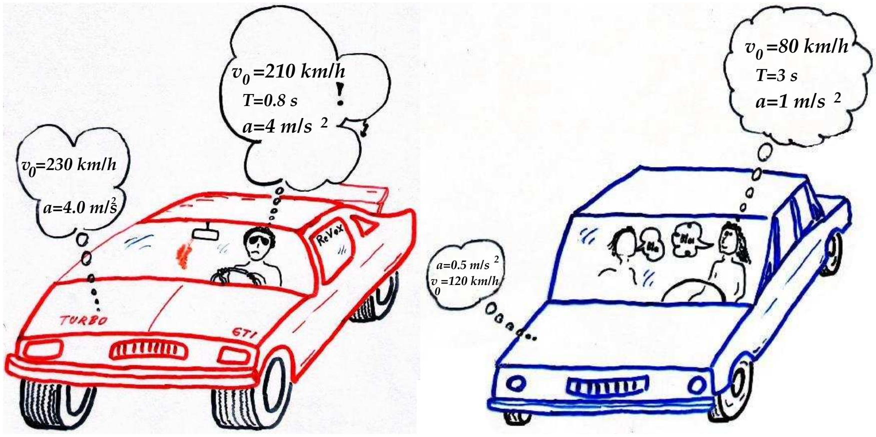



**Fig. 11.2** By using intuitive model parameters like those of Gipps' model or the Intelligent Driver Model (IDM) we can easily model different aspects of the driving behavior (or physical limitations of the vehicle) with corresponding parameter values.

### *11.3.3 Parameters*

We can easily interpret the model parameters by considering the following three standard situations:

- When *accelerating on a free road from a standstill*, the vehicle starts with the maximum acceleration $a$. The acceleration decreases with increasing speed and goes to zero as the speed approaches the desired speed $v_0$. The exponent $\delta$ controls this reduction: The greater its value, the later the reduction of the acceleration when approaching the desired speed. The limit $\delta \to \infty$ corresponds to the acceleration profile of Gipps' model while $\delta = 1$ reproduces the overly smooth acceleration behavior of the Optimal Velocity Model (10.19).
- When *following a leading vehicle*, the distance gap is approximatively given by the *safety distance* $s_0 + vT$ already introduced in Section 11.3.1. The safety distance is determined by the time gap $T$ plus the minimum distance gap $s_0$.
- When *approaching slower or stopped vehicles*, the deceleration usually does not exceed the comfortable deceleration $b$. The acceleration function is smooth during transitions between these situations.

Each parameter describes a well-defined property (Fig. 11.2). For example, transitions between highway and city traffic, can be modeled by solely changing the desired speed (Table 11.2). All other parameters can be kept constant, modeling that somebody who drives aggressively (or defensively) on a highway presumably does so in city traffic as well.

Since the IDM has no explicit reaction time and its driving behavior is given in term of a continuously differentiable acceleration function, the IDM describes more closely the characteristics of semi-automated driving by adaptive cruise con-

**Table 11.2** Model parameters of the Intelligent Driver Model (IDM) and typical values in different scenarios (vehicle length 5 m unless stated otherwise).

| Parameter | Typical Value Highway | Typical Value City Traffic |
|---|---|---|
| Desired speed $v_0$ | 120 km/h | 54 km/h |
| Time gap $T$ | 1.0 s | 1.0 s |
| Minimum gap $s_0$ | 2 m | 2 m |
| Acceleration exponent $\delta$ | 4 | 4 |
| Acceleration $a$ | $1.0\,\mathrm{m/s^2}$ | $1.0\,\mathrm{m/s^2}$ |
| Comfortable deceleration $b$ | $1.5\,\mathrm{m/s^2}$ | $1.5\,\mathrm{m/s^2}$ |

trol (ACC) than that of a human driver. However, it can easily be extended to capture human aspects like estimation errors, reaction times, or looking several vehicles ahead (see Chapter 12).

In contrast to the models discussed previously, the IDM explicitly distinguishes between the safe time gap $T$, the speed adaptation time $\tau = v_0/a$, and the reaction time $T_r$ (zero in the IDM, nonzero in the extension described in Chapter 12). This allows us not only to reflect the conceptual difference between ACCs and human drivers in the model, but also to differentiate between more nuanced driving styles such as "sluggish, yet tailgating" (high value of $\tau = v_0/a$, low value for $T$) or "agile, yet safe driving" (low value of $\tau = v_0/a$, normal value for $T$, low value for $b$).[8] Furthermore, all these driving styles can be adopted independently by ACC systems (reaction time $T_r \approx 0$, original IDM), by attentive drivers ($T_r$ comparatively small, extended IDM), and by sleepy drivers ($T_r$ comparatively large, extended IDM).

### *11.3.4 Intelligent Braking Strategy*

The term $v\Delta v/(2\sqrt{ab})$ in the desired distance $s^*$ (11.15) of the IDM models the dynamical behavior when approaching the leading vehicle. The equilibrium terms $s_0 + vT$ always affect $s^*$ due to the required continuous transitions from and to the equilibrium state. Nevertheless, to study the braking strategy itself, we will set these terms to zero, together with the free acceleration term $a[1-(v/v_0)^\delta]$ of the IDM acceleration equation. When approaching a standing vehicle or a red traffic light ($\Delta v = v$), we then find

$$\dot{v} = -a\left(\frac{s^*}{s}\right)^2 = -\frac{av^2(\Delta v)^2}{4abs^2} = -\left(\frac{v^2}{2s}\right)^2 \frac{1}{b}. \tag{11.16}$$

With the *kinematic deceleration* defined as

[8] Obviously, the first behavior promotes instabilities which will be confirmed by the stability analysis in Chapter 15.

$$b_{\text{kin}} = \frac{v^2}{2s}, \tag{11.17}$$

this part of the acceleration can be written as

$$\dot{v} = -\frac{b_{\text{kin}}^2}{b}. \tag{11.18}$$

When braking with deceleration $b_{\text{kin}}$, the braking distance is exactly the distance to the leading vehicle, thus $b_{\text{kin}}$ is the minimum deceleration required for preventing a collision. With Eq. (11.18), we now understand the self-regulating braking strategy of the IDM:

- A "critical situation" is defined by $b_{\text{kin}}$ being greater than the comfortable deceleration $b$. In such a situation, the actual deceleration is *even stronger* than necessary, $|\dot{v}| = b_{\text{kin}}^2/b > b_{\text{kin}}$. This overcompensation decreases $b_{\text{kin}}$ and thus helps to "regain control" over the situation.
- In a non-critical situation ($b_{\text{kin}} < b$), the actual deceleration is less than the kinematic deceleration, $b_{\text{kin}}^2/b < b_{\text{kin}}$. Thus, $b_{\text{kin}}$ increases in the course of time and approaches the comfortable deceleration.

Hence, the braking strategy is *dynamically self-regulating* towards a situation in which the kinematic deceleration equals the comfortable deceleration. One can show (see Problem 11.4) that this self-regulation is explicitly given by the differential equation

$$\frac{\mathrm{d}b_{\text{kin}}}{\mathrm{d}t} = \frac{v\,b_{\text{kin}}}{s\,b}\left(b - b_{\text{kin}}\right). \tag{11.19}$$

Thus, the kinematic deceleration drifts towards the comfortable deceleration in *any* situation.

In the above considerations, we have ignored parts of the IDM acceleration function. To estimate their effects, the time series of Fig. 11.4(e) display the complete IDM dynamics when approaching an initially very distant, standing obstacle ($b_{\text{kin}} \ll b$): First, the deceleration increases towards the comfortable deceleration according to Eq. (11.19). However, due to the defensive nature of the neglected terms, the comfortable value is never realized, at least for the first vehicle. Eventually, the deceleration smoothly reduces until the vehicle stops with exactly the minimum gap $s_0$ left between itself and the obstacle. The following vehicles experience slightly larger decelerations than the comfortable ones, but without having to perform any emergency braking or being in danger of a collision.

Figure 11.3 shows the effects of the self-regulatory braking strategy in a situation where the vehicle is suddenly forced to stop. Drivers with $b = 1\,\mathrm{m/s^2}$ will perceive this situation as "critical" ($b_{\text{kin}} = v^2/(2s) = 1.9\,\mathrm{m/s^2}$) and overcompensate with even stronger deceleration. In contrast, if the comfortable deceleration is given by $b = 4\,\mathrm{m/s^2}$, the comfortable deceleration is initially well above the kinematic deceleration and the simulated driver will brake only weakly, so that $b_{\text{kin}}$ increases. Again, due to the other terms in the acceleration function, the actual deceleration will not reach the value of comfortable deceleration.

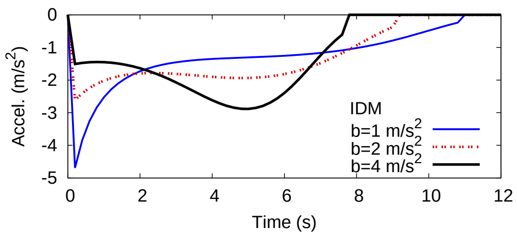



**Fig. 11.3** Acceleration time-series of approaching the stop line of a red traffic light for different values of the comfortable deceleration. The initial speed is $v = 54\,\mathrm{km/h}$. The traffic light switches to red (at time $t = 0$) when the vehicle is 60 m away.

Why do “IDM drivers” act in a more anticipatory manner for smaller values of $b$? Yet why are very small values of $b$ (less than about $1\,\mathrm{m/s^2}$) not meaningful?

Consider the situation of approaching a standing obstacle as described above and convince yourself that the effect of the dynamical part of $s^*$ on the acceleration prevails against all other terms. Furthermore, show that these other terms are negative in nearly all situations, thus making the driving behavior more defensive.

### *11.3.5 Dynamical Properties*

The *fact sheet* of the IDM, Fig. 11.4, shows IDM simulations of the two standard scenarios “traffic breakdown at a highway on-ramp” and “acceleration and stopping of a vehicle platoon in city traffic”.

**Highway traffic.** The speed field in the highway scenario (Fig. 11.4(a)) exhibits dynamics similar to the one found in Gipps’ model (cf. Fig. 11.1): Stationary congested traffic is found close to the bottleneck, while, further upstream, stop-and-go waves emerge and travel upstream with a velocity of approximately $-15\,\mathrm{km/h}$. The wavelength tends to be smaller than in real stop-and-go traffic, but the empirical spatiotemporal dynamics are otherwise reproduced very well. The growing stop-and-go waves in the simulations are caused by a collective instability called *string instability* which will be discussed in more detail in Chapter 15. As we will see in this chapter, the IDM is either unstable with respect to stop-and-go waves (string-unstable) or absolutely stable, depending on the parameters and traffic density. The model is free of accidents, however, except for very unrealistic parameters under specific circumstances.

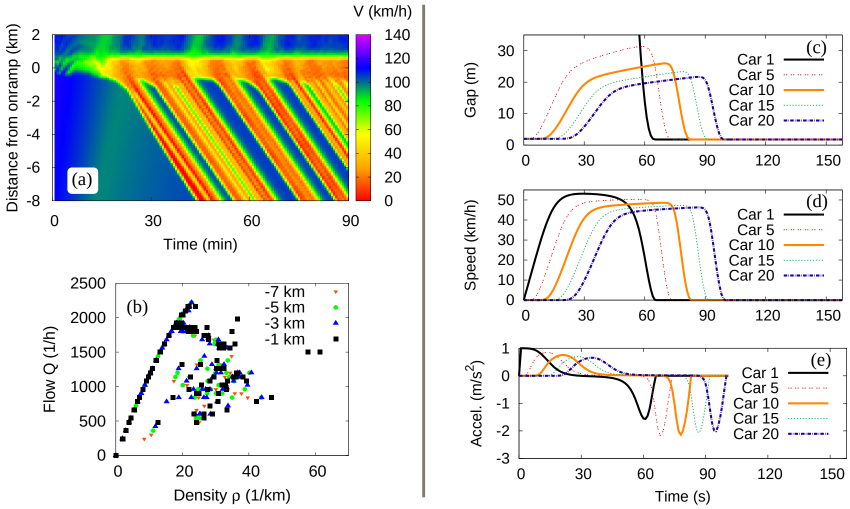



**Fig. 11.4** Fact sheet of the Intelligent Driver Model (11.14). The two standard scenarios "highway" (left) and "city traffic" (right) are simulated with parameters as listed in Table 11.2. See Section 10.5 for a detailed description of the scenarios.

The flow-density diagram of virtual loop detectors in Fig. 11.4(b) reproduces typical aspects of empirical flow-density data:

- Data points representing free traffic fall on a line, while data points from congested traffic are widely scattered.
- The free-traffic branch is not a perfectly straight line but is slightly curved, especially towards the maximum flow.
- Near the maximum flow, the points are arranged in a pattern that looks like a mirror image of the Greek letter $\lambda$ (*inverse-$\lambda$ form*), meaning that for a range of densities (here $\approx 18-25\,\mathrm{veh/h}$), both free and congested traffic states are possible. Thus, the IDM reproduces the empirically observed bistability and the resulting hysteresis effects like the *capacity drop* (about $300\,\mathrm{veh/h}$ or $15\,\%$ in the present example).

Comparing the virtual detector data with the real data in Fig. 11.5(a), (c), and (d), we find almost quantitative agreement of the flow-density, speed-density, and speed-flow diagrams. Contrary to Gipps' model, the IDM also reproduces the curvature of the free-traffic branch correctly. This agreement with the data allows us to scrutinize the nature of the observed strong scattering of flow-density data points corresponding to congested traffic. First, we compare the *estimated* density using the virtual stationary detector data with the *real* spatial density which, of course, is available in the simulation. The result displayed in Fig. 11.5(b) reminds us that one has to be very careful when interpreting flow-density data. Moreover, even the scattering of the data points itself is a matter of the way the data is plotted: While the congested traffic data is much more scattered than the free-flow data in the flow-density data in Fig. 11.5(a), both branches show similar scattering in the speed-flow diagram 11.5(d) – in spite of the fact that both diagrams show the *same* data.

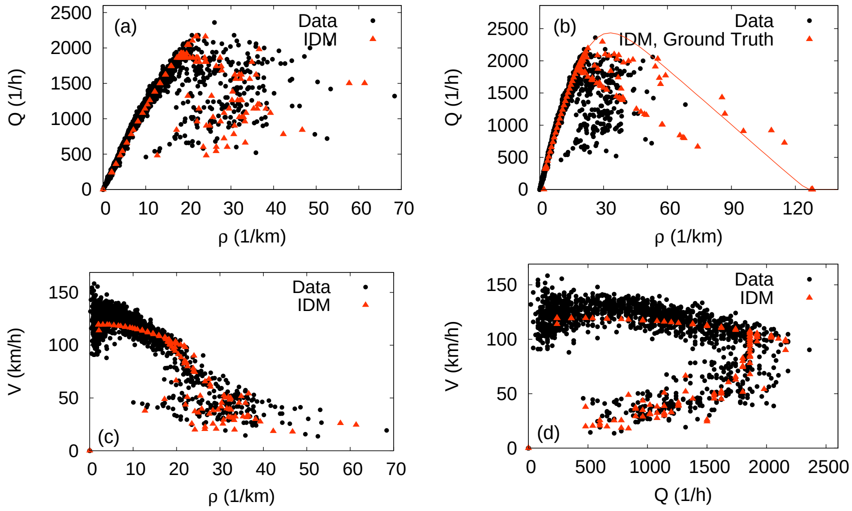



**Fig. 11.5** (a) Fundamental diagram, (c) speed-density diagram, and (d) speed-flow diagram showing data from a virtual detector in the highway simulation shown in Fig. 11.4 (positioned 1 km upstream of the ramp). For comparison, empirical data from a real detector on the Autobahn A5 near Frankfurt, Germany, is shown. Velocities have been calculated using arithmetic means in both the real data and the simulation data. (b) Flow-density diagram with the same empirical data but using the real (local) density for the IDM simulation rather than the density derived from the virtual detectors.

**City traffic.** In the city traffic simulation (Fig. 11.4(c)-(e)) we see a smooth, realistic acceleration/deceleration profile, except in vehicle platoons with speed close to $v_0$ where followers do not accelerate up to the desired speed and thus the distance between the vehicles does not reach a constant value before the braking maneuver begins. This happens because, when approaching the desired speed, the free acceleration function decreases continuously to zero while the interaction (braking) term $s^*/s$ remains finite (reaching zero only in the limit $s \to \infty$). Thus, for $v \lesssim v_0$, the actual steady-state equilibrium distance (where the free acceleration and the interaction terms cancel each other) is significantly larger than $s^*(v,0)$. In the next section, we will investigate this more closely and propose a solution in Section 11.3.7.

### *11.3.6 Steady-State Equilibrium*

By postulating $\dot{v} = \Delta v = 0$ we obtain the condition for the steady-state equilibrium of the IDM from the acceleration function (11.14):

$$1 - \left(\frac{v}{v_0}\right)^{\delta} - \left(\frac{s_0 + vT}{s}\right)^2 = 0. \qquad (11.20)$$

For arbitrary values of $\delta$ we can solve this equation in closed form only for $s$ (cf. Fig. 11.6),

$$s = s_e(v) = \frac{s_0 + vT}{\sqrt{1 - \left(\frac{v}{v_0}\right)^{\delta}}}. \qquad (11.21)$$

This yields the equilibrium gap $s_e(v)$ with the speed being the independent variable

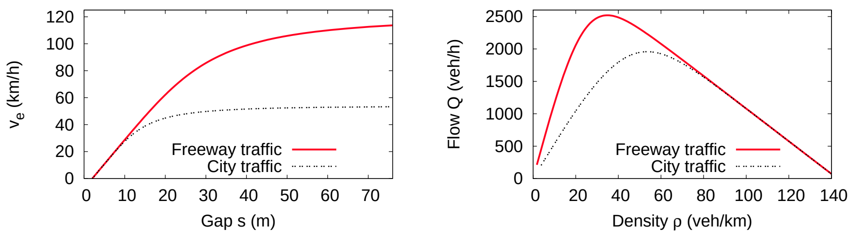



**Fig. 11.6** Microscopic (left) and macroscopic (right) fundamental diagram of the IDM using the parameters shown in Table 11.2.

(instead of the equilibrium speed $v_e(s)$ as a function of the gap). Using the micro-macro relation (10.16), $s_e = \frac{1}{\rho_e} - 1$, $v = V$, and $Q_e = \rho_e V$, we obtain the speed-density and the fundamental diagrams shown in Fig. 11.6.

Note that due to the continuous transition between free and congested traffic, the equilibrium gap $s_e(v)$ is *not* given by $s^*(v,0) = s_0 + vT$. Instead, for $v \lesssim v_0$ it is much larger which can be seen by looking at the denominator of Eq. (11.21). Therefore the fundamental diagram is not a perfect triangle but rounded close to the maximum flow. This causes the curvature in the macroscopic speed-density and flow-density diagrams (Fig. 11.5), but also produces the mentioned unrealistic car-following behavior in platoons with identical driver-vehicle units.

### *11.3.7 Improved Acceleration Function*

Using the IDM as example, this section shows the scientific modeling process, aiming at eliminating some deficiencies of a model while retaining the good and well-tested features and keeping the model parsimonious, i.e., adding as few model parameters as possible.[9] The IDM is unrealistic in following aspects:

[9] Ideally, no parameters are added as in this example.

- If the actual speed exceeds the desired speed (e.g., after entering a zone with a reduced speed limit), the deceleration is unrealistically large, particularly for large values of the acceleration exponent $\delta$.
- Near the desired speed $v_0$, the steady-state gap (11.21) becomes much greater than $s^*(v,0) = s_0 + vT$ so that the model parameter $T$ loses its meaning as the desired time gap. This means that a platoon of identical drivers and vehicles disperses much more than observed. Moreover, not all cars will reach the desired speed (Fig. 11.4(c,d)).
- If the actual gap is considerably smaller than desired (which may happen if another vehicle cuts too close when changing lanes) the braking reaction to regain the desired gap is exaggerated as illustrated in Problem 11.3.

We will treat the first two aspects here while the third aspect (which is only relevant in multi-lane situations) will be deferred to Section 11.3.8.

To improve the behavior for $v > v_0$, we require that the maximum deceleration must not exceed the comfortable deceleration $b$ if there are no interactions with other vehicles or obstacles. The parameter $\delta$ should retain its meaning also in the new regime, i.e., leading to smooth decelerations to the new desired speed for low values and decelerating more "robotically" for high values. Furthermore, the free acceleration function $a_{\text{free}}(v)$ should be continuously differentiable, and remain unchanged for $v \le v_0$, i.e., $a_{\text{free}}(v) = \lim_{s\to\infty} a_{\text{IDM}}(s,v,\Delta v)$ for $v \le v_0$. Probably the simplest free-acceleration function meeting these conditions is given by

$$a_{\text{free}}(v) = \begin{cases} a\left[1 - \left(\frac{v}{v_0}\right)^{\delta}\right] & \text{if } v \le v_0, \\ -b\left[1 - \left(\frac{v_0}{v}\right)^{a\delta/b}\right] & \text{if } \ v > v_0. \end{cases} \tag{11.22}$$

To improve the behavior near the desired speed, we tighten the second condition in Section 11.3.1 by requiring that the equilibrium gap $s_e(v) = s^*(v,0)$ should be *strictly equal* to $s_0 + vT$ for $v < v_0$. However, we would like to implement any modification as conservatively as possible in order to preserve all the other meaningful properties of the IDM (especially the "intelligent" braking strategy). Thus, changes should only have an effect

- near the steady-state equilibrium, i.e., if $z(s,v,\Delta v) = s^*(v,\Delta v)/s \approx 1$,[10]
- and when driving with $v \approx v_0$ and $v > v_0$.

We can accomplish this by distinguishing between the cases $z = s^*(v,\Delta v)/s < 1$ (the actual gap is greater than the desired gap) and $z \ge 1$. The new condition requires $\tilde{a}_{\text{mic}} = 0$ for all input values that satisfy $z(s,v,\Delta v) = 1$ and $v < v_0$. The other conditions in Section 11.3.1 and the conditions (11.1) – (11.5) are automatically satisfied if $\partial\tilde{a}_{\text{mic}}/\partial z < 0$, and if $\tilde{a}_{\text{mic}}(z)$ is continuously differentiable at the transition point $z = 1$. Probably the simplest acceleration function that fulfills all these conditions for $v \le v_0$ is given by

[10] In fact, this condition is more general since it also includes continuations of the steady state to nonstationary situations.

$$\left.\frac{dv}{dt}\right|_{v \le v_0} = \begin{cases} a\,(1-z^2) & z = \frac{s^*(v,\Delta v)}{s} \ge 1, \\ a_{\text{free}} \left(1 - z^{(2a)/a_{\text{free}}}\right) & \text{otherwise.} \end{cases} \tag{11.23}$$

For $v > v_0$, there is no steady-state following distance, and we simply combine the free acceleration $a_{\text{free}}$ and the interaction acceleration $a(1-z^2)$ such that the interaction vanishes for $z \le 1$ and the resulting acceleration function is continuously differentiable:[11]

$$\left.\frac{dv}{dt}\right|_{v > v_0} = \begin{cases} a_{\text{free}} + a(1-z^2) & z(v,\Delta v) \ge 1, \\ a_{\text{free}} & \text{otherwise.} \end{cases} \tag{11.24}$$

This *Improved Intelligent Driver Model* (IIDM) uses the same set of model parameters as the IDM and produces essentially the same behavior except when vehicles follow each other near the desired speed or when the vehicle is faster than the desired speed. Simulating the standard city traffic scenario with the IIDM shows that all vehicles in the platoon now accelerate up to the desired speed (Fig. 11.7, left) while the self-stabilizing braking strategy and the observance of a comfortable deceleration are still in effect (Fig. 11.7, right).

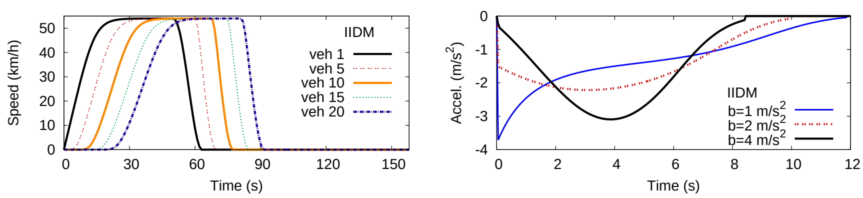



**Fig. 11.7** Simulation of the city traffic scenario (left) and the situation shown in Fig. 11.3 (right) using the Improved Intelligent Driver Model (IIDM) with the parameters listed in Table 11.2.

However, the fundamental diagram is an exact triangle now. Thus, simulating highway traffic will no longer produce curved free-traffic branches in the flow-density, speed-flow, and speed-density diagrams (contrary to the unmodified IDM, cf. Fig. 11.5). Problem 11.6 discusses an alternative cause of this curvature.

### *11.3.8 Model for Adaptive Cruise Control*

While ACC systems only automate the longitudinal driving task, they must also react reasonably if the sensor input variables – the gap $s$ and approaching rate $\Delta v$ – change discontinuously as a consequence of "passive" lane changes (another lane-

[11] At $v = v_0$ and $z > 1$, the full acceleration function (11.23), (11.24) is only continuous, but not differentiable with respect to $v$. It would require a disproportionate amount of complication to resolve this special case of little relevance.

changing vehicle becomes the new leader), and also "active" lane changes (the ACC driver changes lanes manually). This means, the third of the IDM deficiencies mentioned in the previous Section 11.3.7 – overreactions when the gap decreases discontinuously by external actions – must be taken care of.

The reason for the overreactions is the intention of the IDM (and IIDM) development to be accident-free even in the *worst case*, in which the driver of the leading vehicle suddenly brakes to a complete standstill. However, there are situations characterized by low speed differences and small gaps where human drivers rely on the fact that the drivers of preceding vehicles will *not* suddenly initiate full-stop emergency brakings. In fact, they consider such situations only as mildly critical. As a consequence, a more plausible and realistic driving behavior will result when drivers act according to the *constant-acceleration heuristic* (CAH) rather than considering the worst-case scenario. The CAH is based on the following assumptions:

- The accelerations of the considered and leading vehicle will not change in the near future (generally a few seconds).
- No safe time gap or minimum distance is required at any given moment.
- Drivers (or ACC systems) react without delay, i.e., with zero reaction time.

For actual values of the gap $s$, speed $v$, speed $v_l$ of the leading vehicle, and constant accelerations $\dot{v}$ and $\dot{v}_l$ of both vehicles, the maximum acceleration $\max(\dot{v}) = a_{\text{CAH}}$ that does not lead to an accident under the CAH assumption is given by

$$a_{\text{CAH}}(s, v, v_l, \dot{v}_l) = \begin{cases} \frac{v^2 \tilde{a}_l}{v_l^2 - 2s\tilde{a}_l} & \text{if } \; v_l(v - v_l) \le -2s\tilde{a}_l, \\ \tilde{a}_l - \frac{(v-v_l)^2 \Theta(v-v_l)}{2s} & \text{otherwise.} \end{cases} \tag{11.25}$$

The effective acceleration $\tilde{a}_l(\dot{v}_l) = \min(\dot{v}_l, a)$ (with the maximum acceleration parameter $a$) has been used to avoid the situation where leading vehicles with higher acceleration capabilities may cause "drag-along effects" of the form (11.6), or other artefacts violating the general plausibility conditions (11.1) – (11.5). The condition $v_l(v - v_l) = v_l \Delta v \le -2s\dot{v}_l$ is true if the vehicles have stopped at the time the minimum gap $s = 0$ is reached. The Heaviside step function $\Theta(x)$ (with $\Theta(x) = 1$ if $x \ge 0$, and zero, otherwise) eliminates negative approaching rates $\Delta v$ for the case that both vehicles are moving at the time $t^*$ of least distance. Otherwise, $t^*$ would lie in the past.

In order to retain all the "good" properties of the IDM, we will use the CAH acceleration (11.25) only as an *indicator* to determine whether the IDM will lead to unrealistically high decelerations, and modify the acceleration function of a model for ACC vehicles only in this case. Specifically, the proposed ACC model is based on following assumptions:

- The ACC acceleration is never lower than that of the IIDM. This is motivated by the circumstance that the IDM and the IIDM are accident-free, i.e., sufficiently defensive.
- If both, the IIDM and the CAH, produce the same acceleration, the ACC acceleration is the same as well.

- If the IIDM produces extreme decelerations, while the CAH yields accelerations in the comfortable range (greater than $-b$), the situation is considered to be mildly critical, and the resulting acceleration should be between $a_{\mathrm{CAH}} - b$ and $a_{\mathrm{CAH}}$. Only for *very* small gaps, the decelerations should be somewhat higher to avoid an overly reckless driving style.
- If both, the IIDM and the CAH, result in accelerations significantly below $-b$, the situation is seriously critical and the ACC acceleration is given by the maximum of the IIDM and CAH accelerations.
- The ACC acceleration should be a continuous and differentiable function of the IIDM and CAH accelerations. Furthermore, it should meet the consistency requirements (11.1) – (11.5).

Probably the most simple functional form satisfying these criteria is given by

$$a_{\mathrm{ACC}} = \begin{cases} a_{\mathrm{IIDM}} & a_{\mathrm{IIDM}} \geq a_{\mathrm{CAH}}, \\ (1-c)\, a_{\mathrm{IIDM}} + c \left[ a_{\mathrm{CAH}} + b \tanh\left( \frac{a_{\mathrm{IIDM}} - a_{\mathrm{CAH}}}{b} \right) \right] & \text{otherwise.} \end{cases} \tag{11.26}$$

This *ACC model* has only one additional parameter compared to the IDM/IIDM, the "coolness factor" $c$. For $c = 0$ one recovers the IIDM while $c = 1$ corresponds to the "pure" ACC model. Since the pure ACC model would produce a reckless driving behavior for very small gaps, a small fraction $1 - c$ of the IIDM is added. It turns out that a contribution of 1 % (corresponding to $c = 0.99$) gives a good compromise between reckless and overly timid behavior in this situation while it is essentially irrelevant, otherwise.

In contrast to the other models of this section, the ACC model has the *acceleration* $\dot{v}_l$ of the leading vehicle as additional exogenous factor (besides the speed, the speed difference, and the gap). This models a behavior similar to the human reactions to brake lights, but in a continuous rather than in an off-on way.[12]

The Figs. 11.8 and 11.9 show the effect of this model improvement: In the mildly critical situation of Fig. 11.8, a lane-changing car driving at the same speed as the considered car cuts in front leaving a gap of only 10 m which is less than one third of the "safe" gap $s_0 + vT = 35.3\,\mathrm{m}$. While the IIDM (and IDM) will initiate a short emergency braking maneuver in this situation, the ACC reflects a relaxed reaction by braking at about the comfortable deceleration. In contrast, if the situation becomes really critical (Fig. 11.9), both the (I)IDM and the ACC model will initiate an emergency braking.

When implementing all these modifications, one needs to bear in mind that, in most situations, the ACC model should behave very similarly to the IDM so as to retain its well-tested good properties. To verify this, Fig. 11.10 shows the familiar *fact sheet* for the two standard situations. As expected, there is little difference compared to the IDM (and the IIDM) since the modified behavior kicks in only in the highway scenario, and only at the merging region of the on-ramp.

[12] Since the acceleration $\dot{v}_l$ cannot be measured directly, it is obtained by numerical differentiation of the approaching rate and the speed-changing rate. Care has to be taken to control the resulting discretization errors.

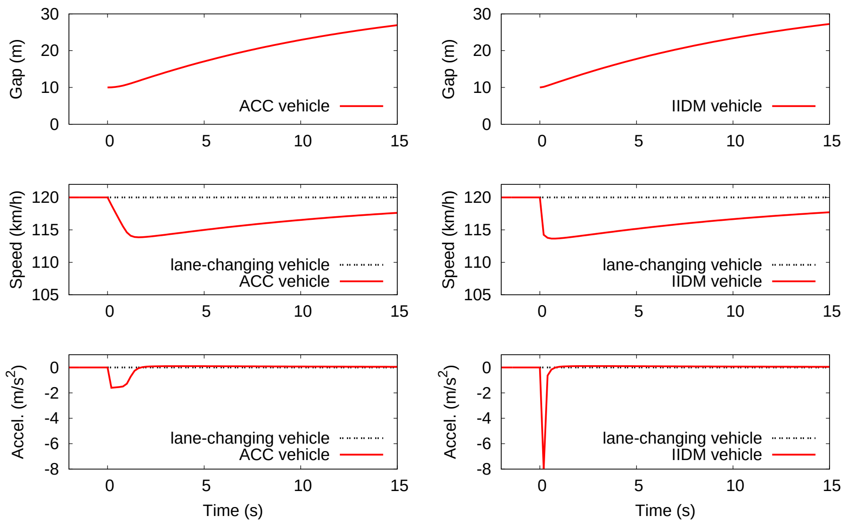



**Fig. 11.8** Response of an ACC and an IIDM vehicle (parameters of Table 11.2 and coolness factor $c = 0.99$) to the lane-changing maneuver of another vehicle immediately in front of the considered vehicle. The initial speed of both vehicles is 120 km/h (equal to the desired speed), and the initial gap is 10 m which is about 30 % of the desired gap. This can be considered as a "mildly critical" situation.

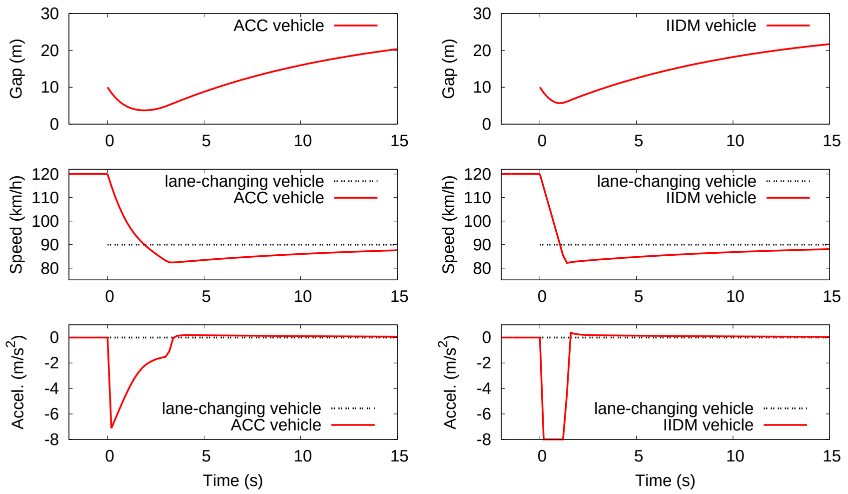



**Fig. 11.9** Response of an ACC and an IIDM vehicle to a dangerous lane-changing maneuver of a slower vehicle (speed 90 km/h) immediately in front of the considered vehicle driving initially at $v_0 = 120\,\mathrm{km/h}$. The decelerations are restricted to $8\,\mathrm{m/s}^2$. The parameters are according to Table 11.2 and a coolness factor of $c = 0.99$ for the ACC vehicles. The desired speed of the lane-changing vehicle is reduced to 90 km/h.

In summary, the ACC model can be considered as a minimal fully operative control model for ACC systems. With minor modifications, it has been implemented in real cars and tested on test tracks as well as on public roads and highways.

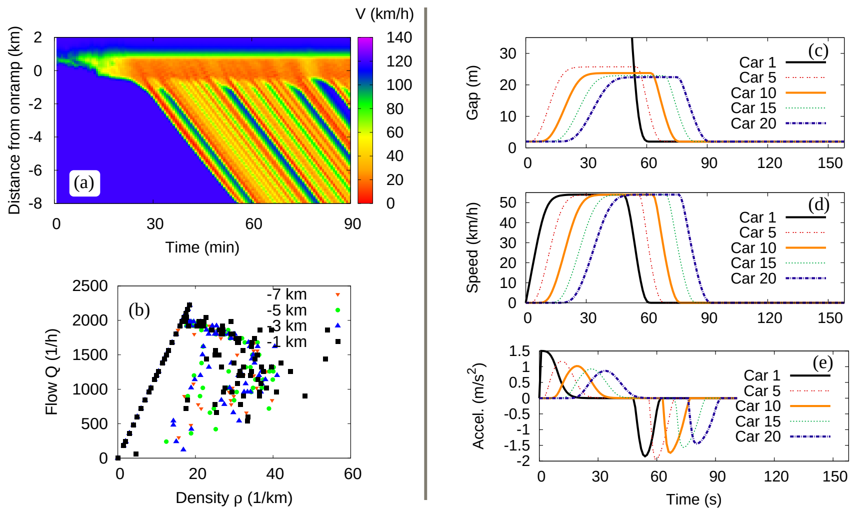



**Fig. 11.10** Fact sheet of the ACC-model. The two standard scenarios "highway" (left) and "city traffic" (right) are simulated with the parameter values listed in Table 11.2 and the "coolness factor" $c = 0.99$. See Section 10.5 for a detailed description of the simulation scenarios.

## Problems

### 11.1. Conditions for the microscopic fundamental diagram

Use the consistency conditions (11.1) – (11.5) to derive the conditions (11.7) that have to be fulfilled by the steady-state speed-distance relation $v_e(s)$ (microscopic fundamental diagram).

### 11.2. Rules of thumb for the safe gap and braking distance

1. A common US rule for the safe gap is the following: "Leave one car length for every ten miles per hour of speed". Another rule says "Leave a time gap of two seconds". Compare these two rules assuming a typical car length of 15 ft. For which car length are both rules equivalent?
2. In Continental European countries, one learns in driving schools the following rule: "The safe gap should be at least half the reading of the speedometer". Translate this rule into a safe time gap rule and compare it with the US rule stated above. Take into account that, in Continental Europe, speed is commonly expressed in terms of km per hour.

3. A rule of thumb for the braking distance says "Speed squared and divided by 100". If speed is measured in km/h, what braking deceleration is assumed by this rule?

**11.3. Reaction to vehicles merging into the lane**
A vehicle enters the lane of the considered car causing the gap $s$ to fall short of the equilibrium gap $s_e$ by 50 %. Both vehicles drive at the same speed. Find the resulting (negative) accelerations produced by the simplified Gipps' model and the IDM, assuming the parameter values $\Delta t = 1\,\mathrm{s}$, $b = 2\,\mathrm{m/s^2}$, $a = 1\,\mathrm{m/s^2}$, $\delta = 4$, and $v = v_0/2 = 72\,\mathrm{km/h}$ for all vehicles. (No other parameter values are needed for this problem.)

**11.4. The IDM braking strategy**
Derive Eq. (11.19) for the explicit description of the self-regulating braking strategy when approaching a standing obstacle ($\Delta v = v$). Assume that the IDM acceleration can be reduced to the braking term $\dot{v} = -b_{\mathrm{kin}}^2/b$ for this case. *Hint*: Keep in mind that $\dot{s} = -\Delta v$.

**11.5. Analysis of a microscopic model**
Assume a car-following model that is given by the following acceleration equation:

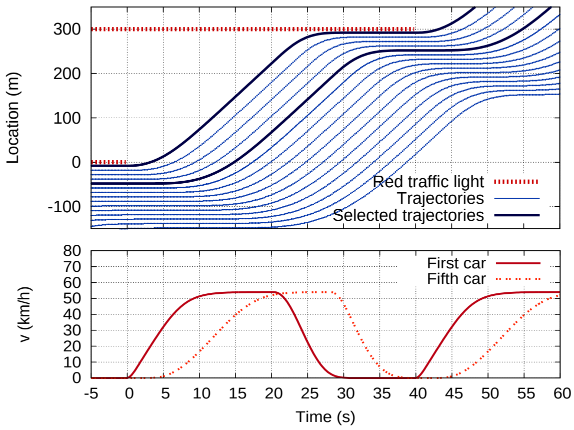
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \begin{cases} a & \text{if } v < \min(v_0, v_{\mathrm{safe}}), \\ 0 & \text{if } v = \min(v_0, v_{\mathrm{safe}}), \\ -a & \text{otherwise,} \end{cases} \qquad v_{\mathrm{safe}} = -aT + \sqrt{a^2T^2 + v_l^2 + 2a(s - s_0)}.$$

As usual, $v_l$ is the speed of the leading vehicle, and $s$ the corresponding bumper-to-bumper gap.

1. Explain the meaning of the parameters $a$, $s_0$, $v_0$, and $T$ by examining (i) the acceleration on a free road segment, (ii) the driving behavior when following another vehicle with constant speed and gap, and (iii) the braking maneuver performed when approaching a standing vehicle.
2. Find the steady-state speed $v_e(s)$ as a function of the distance assuming $v_0 = 20\,\mathrm{m/s}$, $a = 1\,\mathrm{m/s^2}$, $T = 1.6\,\mathrm{s}$, and $s_0 = 3\,\mathrm{m}$. Also, sketch the fundamental diagram for vehicles of length 5 m.
3. Assume that a vehicle standing at position $x = 0$ for $t \le 0$ accelerates for $t > 0$ and then stops at a red traffic light at $x = 603\,\mathrm{m}$. Derive the speed function $v(t)$ for this scenario, assuming the parameter values $v_0 = 20\,\mathrm{m/s}$, $a = 1\,\mathrm{m/s^2}$, $T = 0$, and $s_0 = 3\,\mathrm{m}$. (Hints: The traffic light is modeled by a standing "virtual" vehicle; the vehicle will reach its desired speed in this scenario.)

**11.6. Heterogeneous traffic**
For identical vehicles and drivers, the modified IDM with its strictly triangular fundamental diagram (IIDM) does not produce the pre-breakdown speed drop observed in Fig. 11.5. Is it possible to produce the speed drop by introducing a combination of different desired speeds or the possibility of passing maneuvers?

**11.7. City traffic in the modified IDM (IIDM)**
On a road segment with two traffic lights, a number of vehicles is standing in front of the first traffic light. When the light turns green, the vehicles accelerate but have to stop again at the second traffic light. The upper panel shows trajectories of all 15 vehicles and the red lights (horizontal lines). The lower panel shows the corresponding speeds of the two bold trajectories.




1. Estimate the capacity $C$ of the free road segment (without traffic lights) by finding the maximum possible flow.
2. How many vehicles are able to pass the traffic light at $x = 0$ if the green light is on for (i) 5 s, (ii) 15 s, or (iii) 40 s? Find appropriate $\tau_0$ and $\beta$ such that $\tau(n) = \tau_0 + \beta n$ is the time the light has to be green to let $n$ vehicles pass.
3. Estimate the velocity $c_{\text{cong}}$ of the transition "standing traffic" $\rightarrow$ "starting to move" from the shown trajectories.
4. Estimate the IDM parameters $v_0$, $l_{\text{eff}} = l + s_0$, $T = 1/(\rho_{\max} c_{\text{cong}})$, $a$, and $b$ used in the simulation. Add appropriate tangents to the speed diagram to find the accelerations.

## Further Reading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- Krauss, S.: Microscopic Modeling of Traffic Flow: Investigation of Collision Free Vehicle Dynamics. Ph.D. Thesis, University of Cologne, Cologne, Germany (1997)
- Treiber, M., Hennecke, A., Helbing, D.: Congested traffic states in empirical observations and microscopic simulations. Physical Review E **62** (2000) 1805–1824

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*Subproblem 3 (emergency braking).* At first, we determine the initial distance such that a driver driving at $v_1 = 50\,\mathrm{km/h}$ just manages to stop before hitting the child:

$$s(0) = s_{\mathrm{stop}}(v_1) = 25.95\,\mathrm{m}.$$

Now we consider a speed $v_2 = 70\,\mathrm{km/h}$ but the same initial distance $s(0) = 25.95\,\mathrm{m}$ as calculated above. At the end of the reaction time, the child is just

$$s(T_r) = s(0) - v_2 T_r = 6.50\,\mathrm{m}$$

away from the front bumper. Now, the driver would need the additional braking distance $s_B(v_2) = 23.6\,\mathrm{m}$ for a complete stop. However, only $6.50\,\mathrm{m}$ are available resulting in a difference $\Delta s = 17.13\,\mathrm{m}$. With this information, the speed at collision can be calculated by solving $\Delta s = (\Delta s)_B(v) = v^2/(2b_{\mathrm{max}})$ for $v$, i.e.,

$$v_{\mathrm{coll}} = \sqrt{2b_{\mathrm{max}}\Delta s} = 16.56\,\mathrm{m/s} = 59.6\,\mathrm{km/h}.$$

*Remark:* This problem stems from a multiple-choice question of the theoretical exam for a German driver's licence. The official answer is 60 km/h.

## Problems of Chapter 11

**11.1 Conditions for the microscopic fundamental diagram.** The plausibility condition (11.5) is valid for any speed $v_l$ of the leading vehicle. This also includes standing vehicles where Eq. (11.5) becomes $a_{\mathrm{mic}}(s,0,0) = 0$ for $s \le s_0$. This corresponds to the steady-state condition $v_e(s) = 0$ for $s \le s_0$.

Conditions (11.1) and (11.2) are valid for any speed $v_l$ of the leader as well, including the steady-state situation $v_l = v$ or $\Delta v = 0$. For the alternative acceleration function $\tilde{a}(s,v,\Delta v)$, this means

$$\frac{\partial \tilde{a}(s,v,0)}{\partial s} \ge 0, \quad \frac{\partial \tilde{a}(s,v,0)}{\partial v} < 0.$$

Along the one-dimensional manifold of steady-state solutions $\{v_e(s)\}$ for $s \in [0,\infty[$, we have $\tilde{a}(s,v_e(s),0) = 0$, so the differential change $\mathrm{d}\tilde{a}$ along the equilibrium curve $v_e(s)$ must vanish as well:

$$\mathrm{d}\tilde{a} = \frac{\partial \tilde{a}(s,v_e(s),0)}{\partial s}\mathrm{d}s + \frac{\partial \tilde{a}(s,v_e(s),0)}{\partial v} v_e'(s)\mathrm{d}s = 0,$$

hence

$$v_e'(s) = \frac{-\partial \tilde{a}(s,v,0)/\partial s}{\partial \tilde{a}(s,v,0)/\partial v} \ge 0.$$

If the leading vehicle is outside the interaction range, we have $v_e'(s) = 0$ (second condition of Eq. (11.2)). Finally, the condition $\lim_{s\to\infty} v_e(s) = v_0$ follows directly from the second part of condition (11.1).

### 11.2 Rules of thumb for the safe gap and braking distance

*Subproblem 1.* One mile corresponds to 1.609 km. However, the US rule does not give explicit values for a vehicle length. Here, we assume $15\,\text{ft} = 4.572\,\text{m}$. In any case, the gap $s$ increases linearly with the speed $v$, so the time gap $T = s/v$ is independent of speed. Implementing this rule, we obtain

$$T = \frac{s}{v} = \frac{15\,\text{ft}}{10\,\text{mph}} = \frac{4.572\,\text{m}}{16.09\,\text{km/h}} = \frac{4.572\,\text{m}}{4.469\,\text{m/s}} = 1.0\,\text{s}.$$

Notice that, in the final result, we rounded off generously. After all, this is a rule of thumb and more significant digits would feign a non-existent precision.[7] Notice that this rule is consistent with typically observed gaps (cf. Fig. 4.8).

*Subproblem 2.* Here, the speedometer reading is in units of km/h, and the space gap is in units of meters. Again, the quotient, i.e., the time gap $T$ is constant and given by (watch out for the units)

$$T = \frac{s}{v} = \frac{\frac{1}{2}\text{m}\left(\frac{v}{\text{km/h}}\right)}{v} = \frac{\frac{1}{2}\text{m}}{\text{km/h}} = \frac{0.5\,\text{h}}{1\,000} = \frac{1\,800\,\text{s}}{1\,000} = 1.8\,\text{s}.$$

*Subproblem 3.* The kinematic *braking distance* is $s(v) = v^2/(2b)$, so the cited rule of thumb implies that the braking deceleration does not depend on speed. By solving the kinematic braking distance for $b$ and inserting the rule, we obtain (again, watch out for the units)

$$b = \frac{v^2}{2s} = \frac{v^2}{0.02\,\text{m}}\left(\frac{\text{km}}{\text{h}\,v}\right)^2 = \frac{50}{3.6^2}\text{m/s}^2 = 3.86\,\text{m/s}^2.$$

For reference, comfortable decelerations are below $2\,\text{m/s}^2$ while emergency braking decelerations on dry roads with good grip conditions can be up to $10\,\text{m/s}^2$, about $6\,\text{m/s}^2$ for wet conditions, and less than $2\,\text{m/s}^2$ for icy conditions. This means, the above rule could lead to accidents for icy conditions but is okay, otherwise.

### 11.3 Reaction to vehicles merging into the lane

*Reaction for the IDM.* For $v = v_0/2$, the IDM steady-state space gap reads

$$s_e(v) = \frac{s_0 + vT}{\sqrt{1 - \left(\frac{v}{v_0}\right)^\delta}} = \frac{s_0 + \frac{v_0 T}{2}}{\sqrt{1 - \left(\frac{1}{2}\right)^\delta}}.$$

---

[7] There is also a more conservative variant of this rule where one should leave one car length every five mph corresponding to the "two-second rule" $T = 2.0\,\text{s}$.

The prevailing contribution comes from the prescribed time headway (for $s_0 = 2\,\mathrm{m}$ and $\delta = 4$, the other contributions only make up about 10 %). This problem assumes that the merging vehicle reduces the gap to the considered follower to half the steady-state gap, $s = s_e/2 = v_0 T/4$, while the speed difference remain zero. The new IDM acceleration of the follower (with $a = 1\,\mathrm{m/s^2}$ and $\delta = 4$) is therefore

$$\begin{aligned}
\dot{v}_{\mathrm{IDM}} &= a\left[1 - \left(\frac{v}{v_0}\right)^{\delta} - \left(\frac{s_0 + vT}{s}\right)^2\right] \\
&\overset{(v=v_0/2,\, s=s_e/2)}{=} a\left[1 - \left(\frac{1}{2}\right)^{\delta} - \left(\frac{s_0 + v_0 T/2}{s_e/2}\right)^2\right] \\
&\overset{s_e(v)=s_e(v_0/2)}{=} -3a\left[1 - \left(\frac{1}{2}\right)^{\delta}\right] = -\frac{45}{16}\ \mathrm{m/s^2} = -2.81\,\mathrm{m/s^2}.
\end{aligned}$$

*Reaction for the simplified Gipps' model.* For this model, the steady-state gap in the car-following regime reads $s_e(v) = v\Delta t$. Again, at the time of merging, the merging vehicle has the same speed $v_0/2$ as the follower, and the gap is half the steady-state gap, $s = (v\Delta t)/2 = v_0\Delta t/4$. The new speed of the follower is restricted by the safe speed $v_{\mathrm{safe}}$:

$$v(t+\Delta t) = v_{\mathrm{safe}} = -b\Delta t + \sqrt{b^2(\Delta t)^2 + \left(\frac{v_0}{2}\right)^2 + \frac{b v_0 \Delta t}{2}} = 19.07\,\mathrm{m/s}.$$

This results in an effective acceleration

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_{\mathrm{Gipps}} = \frac{v(t+\Delta t) - v(t)}{\Delta t} \approx -0.93\,\mathrm{m/s^2}.$$

We conclude that the Gipps' model describes a more relaxed driver reaction compared to the IDM. Notice that both the IDM and Gipps' model would generate significantly higher decelerations for the case of slower leading vehicles (dangerous situation).

**11.4 The IDM braking strategy.** A braking strategy is self-regulating if, during the braking process, the *kinematically necessary deceleration* $b_{\mathrm{kin}} = v^2/(2s)$ approaches the comfortable deceleration $b$. In order to show this, we calculate the rate of change of the kinematic deceleration (applying the quotient and chain rules of differentiation when necessary) and set $\dot{s} = -v$ and $\dot{v} = -b_{\mathrm{kin}}^2/b = -v^4/(4bs^2)$, afterwards. This eventually gives Eq. (11.19) of the main text:

$$\begin{aligned}
\frac{\mathrm{d}b_{\mathrm{kin}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2s}\right) = \frac{4vs\dot{v} - 2v^2\dot{s}}{4s^2} \\
&= \frac{v^3}{2s^2}\left(1 - \frac{v^2}{2sb}\right) = \frac{v\,b_{\mathrm{kin}}}{s\,b}\left(b - b_{\mathrm{kin}}\right),
\end{aligned}$$

### 11.5 Analysis of a microscopic model

*Subproblem 1 (parameters).* For interaction-free accelerations, $v_{\text{safe}} > v_0$, so $v_{\text{safe}}$ is not relevant. Hence $v_0$ denotes the desired speed, and $a$ the absolute value of the acceleration and deceleration for the cases $v < v_0$ and $v > v_0$, respectively. The steady-state conditions $s = \text{const.}$ and $v = v_l = v_e = \text{const.}$ give

$$v_e = \min(v_0, v_{\text{safe}}).$$

Without interaction, $v_{\text{safe}} > v_0$, so $v_e = v_0$. With interactions, the safe speed becomes relevant and the above condition yields

$$v_e = v_{\text{safe}} = -aT + \sqrt{a^2T^2 + v_e^2 + 2a(s - s_0)}$$

which can be simplified to

$$s = s_0 + v_e T.$$

Thus, $s_0$ is the minimum gap for $v = 0$, and $T$ the desired time gap. The model produces a deceleration $-a$ not only if $v > v_0$ (driving too fast in free traffic) but also if $v > v_{\text{safe}}$ (driving too fast in congested situations). Furthermore, the model is symmetrical with respect to accelerations and decelerations. Obviously, it is not accident free.

*Subproblem 2 (steady-state speed).* We have already derived the steady-state condition

$$v_e(s) = \min\left(v_0, \frac{s - s_0}{T}\right).$$

Macroscopically, this corresponds to the triangular fundamental diagram

$$Q_e(\rho) = \min\left(v_0\rho, \frac{1 - \rho\, l_{\text{eff}}}{T}\right)$$

where $l_{\text{eff}} = 1/\rho_{\text{max}} = l + s_0$. The capacity per lane is given by $Q_{\text{max}} = (T + l_{\text{eff}}/v_0)^{-1} = 1\,800\,\text{vehicles/h}$ at a density $\rho_{\text{C}} = 1/(l_{\text{eff}} + v_0 T) = 25\,/\text{km}$. For further properties of the triangular fundamental diagram, see Section 8.5.

*Subproblem 3.* The acceleration and braking distances to accelerate from 0 to $20\,\text{m/s}$ or to brake from $20\,\text{m/s}$ to 0, respectively, are the same:

$$s_{\text{a}} = s_b = \frac{v_0^2}{2a} = 200\,\text{m}.$$

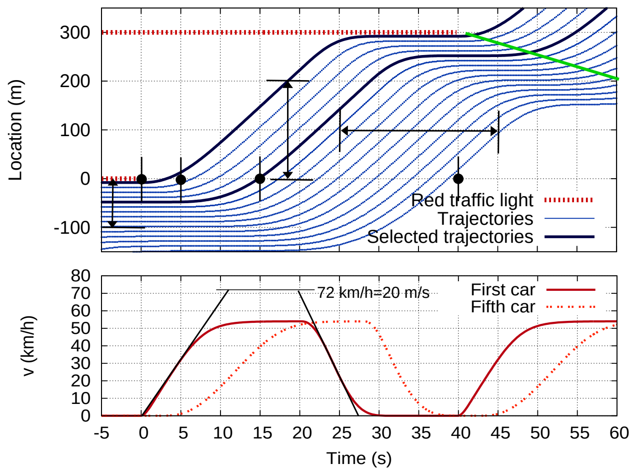
At a minimum gap of $3\,\text{m}$ and the location $x_{\text{stop}} = 603\,\text{m}$ of the stopping line of the traffic light, the acceleration takes place from $x = 0$ to $x_1 = 200\,\text{m}$, and the deceleration from $x_2 = 400\,\text{m}$ to $x_3 = 600\,\text{m}$. The duration of the acceleration and deceleration phases is $v_0/a = 20\,\text{s}$ while the time to cruise the remaining stretch of $200\,\text{m}$ at $v_0$ amounts to $10\,\text{s}$. This completes the information to mathematically describe the trajectory:

$$x(t) = \begin{cases} \frac{1}{2}at^2 & t \le t_1 = 20\,\text{s}, \\ x_1 + v_0(t - t_1) & t_1 < t \le t_2 = 30\,\text{s}, \\ x_2 + v_0(t - t_2) - \frac{1}{2}a(t - t_2)^2 & t_2 < t \le t_3 = 50\,\text{s}, \end{cases}$$

where $t_1 = 20\,\text{s}$, $t_2 = 30\,\text{s}$ and $t_3 = 50\,\text{s}$.

**11.6 Heterogeneous traffic.** The simultaneous effects of heterogeneous traffic and several lanes with lane-changing and overtaking possibilities results in a curved free part of the fundamental diagram even for models that would display a triangular fundamental diagram for identical vehicles and drivers (as the Improved Intelligent Driver Model, IIDM). This can be seen as follows: For heterogeneous traffic, each vehicle-driver class has a different fundamental diagram. Particularly, the density $\rho_C$ at capacity is different for each class, so a simple weighted average of the individual fundamental diagrams would result in a curved free part and a rounded peak. However, without lane-changing and overtaking possibilities, all vehicles would queue up behind the vehicles of the slowest class resulting in a straight free part of the fundamental diagram with the gradient representing the lowest free speed.[8] So, both heterogeneity and overtaking possibilities are necessary to produce a curved free part of the fundamental diagram.

### 11.7 City traffic in the improved IDM




1. For realistic circumstances, the maximum possible flow is given by the *dynamic* capacity, i.e., the outflow from moving downstream congestion fronts. In our case, the “congestion” is formed by the queue of standing vehicles behind a traffic light. Counting the trajectories (horizontal double-arrow in the upper diagram) yields

$$C = Q_{\max} \approx \frac{9\,\text{vehicles}}{20\,\text{s}} = 1\,620\,\text{vehicles/h}.$$

[8] Even when obstructed, drivers can choose their preferred gap (in contrast to the desired speed), so the congested branch of the fundamental diagram is curved even without overtaking possibilities.

2. Counting the trajectories passing $x = 0$ for times less than 5 s, 15 s, and 40 s (black bullets in the upper diagram) gives

$$n(5) = 1,\ n(15) = 5,\ n(40) = 15,$$

respectively. We determine $\beta$ by the average time headway after the first vehicles have passed,

$$\beta = \frac{1}{C} = \frac{40\,\mathrm{s} - 15\,\mathrm{s}}{15-5} = 2.5\,\mathrm{s/vehicles}.$$

We observe, that $\beta$ denotes the inverse of the capacity. The obtained value agrees with the result of the first subproblem within the "measuring uncertainty" of one vehicle.[9] This also gives the *additional time* until the first vehicle passes: $\tau_0 = 15\,\mathrm{s} - 5\beta = 2.5\,\mathrm{s}$. (Notice that this is *not* a reaction time since the IIDM does not have one.)

3. The propagation velocity of the position of the starting vehicles in the queue is read off from the upper diagram:

$$c_{\mathrm{cong}} = -\frac{100\,\mathrm{m}}{20\,\mathrm{s}} = -5\,\mathrm{m/s} = -18\,\mathrm{km/h}.$$

4. We estimate the desired speed by the maximum speed of the speed profile (lower diagram): $v_0 = 15\,\mathrm{m/s} = 54\,\mathrm{km/h}$. The effective length $l_{\mathrm{eff}}$ is equal to the distance between the standing vehicles in the upper diagram: $\rho_{\max} = 1/l_{\mathrm{eff}} = 10\,\mathrm{vehicles}/100\,\mathrm{m} = 100\,\mathrm{vehicles/km}$, i.e., $l_{\mathrm{eff}} = 10\,\mathrm{m}$. Since the steady state of this model corresponds to a triangular fundamental diagram, the time gap parameter $T$ is determined by the propagation speed and the maximum density: $T = -l_{\mathrm{eff}}/c = 2\,\mathrm{s}$. Finally, the maximum acceleration $a$ and the comfortable deceleration $b$ can be read off the lower diagram by estimating the maximum and minimum gradient of the speed profile:

$$a = \frac{20\,\mathrm{m/s}}{10\,\mathrm{s}} = 2\,\mathrm{m/s^2}, \quad b = \frac{20\,\mathrm{m/s}}{7\,\mathrm{s}} = 2.9\,\mathrm{m/s^2}.$$

## Problems of Chapter 12

**12.1 Statistical properties of the Wiener process.** To determine the expectation $\langle w(t)w(t')\rangle$ from the given formal solution $w(t)$ to the stochastic differential equation of the Wiener process, we insert the formal solution into $\langle w(t)w(t')\rangle$ carefully distinguishing the arguments $t$ and $t'$ from the formal integration variables $t_1$ and $t_2$. This gives the double integral

[9] One could have calculated $\beta$ as well using the pairs $\{n(15), n(5)\}$ or $\{n(40), n(5)\}$ with similar results.